Appendix A

Fourier Transform Tables

We here collect several of the Fourier transform pairs developed in the book, including both ordinary and generalized forms. This provides a handy summary and reference and makes explicit several results implicit in the book. We also use the elementary properties of Fourier transforms to extend some of the results.

We begin in Tables A.1 and A.2 with several of the basic transforms derived for the continuous time infinite duration case. Note that both the Dirac delta $\delta(x)$ and the Kronecker delta δ_x appear in the tables. The Kronecker delta is useful if the argument x is continuous or discrete for representations of the form $h(x) = f(x)\delta_x + g(x)(1-\delta_x)$ which means that $h(0) = f(0)$ and $h(x) = g(x)$ when $x \neq 0$.

The transforms in Table A.2 are all obtained from transforms in Table A.1 by the duality property, that is, by reversing the roles of time and frequency.

Several of the previous signals are time-limited (i.e., are infinite duration signals which are nonzero only in a finite interval) and hence have corresponding finite duration signals. The Fourier transforms are the same for any fixed real frequency f, but we have seen that the appropriate frequency domain $\mathcal{S}$ is no longer the real line but only a discrete subset. Table A.3 provides some examples.

Table A.4 collects several discrete time infinite duration transforms. Remember that for these results a difference or sum in the frequency domain is interpreted modulo that domain.

Table A.5 collects some of the more common closed form DFTs.

Table A.6 collects several two-dimensional transforms.

g	$\mathcal{F}(g)$
$\{\sqcap(t);\ t \in \mathcal{R}\}$	$\{\text{sinc}(f); f \in \mathcal{R}\}$
$\{\square_T(t);\ t \in \mathcal{R}\}$	$\{2T\,\text{sinc}(2Tf); f \in \mathcal{R}\}$
$\{e^{-t}u_{-1}(t);\ t \in \mathcal{R}\}$	$\{\frac{1}{1+2\pi i f};\ f \in \mathcal{R}\}$
$\{e^{-\|t\|};\ t \in \mathcal{R}\}$	$\{\frac{2}{1+(2\pi f)^2};\ f \in \mathcal{R}\}$
$\{t \sqcap (t-\frac{1}{2}); t \in \mathcal{R}\}$	$\{\frac{1}{2}\delta_f + \left(\frac{ie^{-i2\pi f}}{2\pi f} + \frac{e^{-i2\pi f}-1}{(2\pi f)^2}\right)(1-\delta_f);\ f \in \mathcal{R}\}$
$\{\wedge(t); t \in \mathcal{R}\}$	$\{\text{sinc}^2(f); f \in \mathcal{R}\}$
$\{e^{-\pi t^2}; t \in \mathcal{R}\}$	$\{e^{-\pi f^2}; f \in \mathcal{R}\}$
$\{e^{+i\pi t^2}; t \in \mathcal{R}\}$	$\{\frac{1}{\sqrt{-i}}e^{-i2\pi f^2}; f \in \mathcal{R}\}$
$\{\text{sgn}(t); t \in \mathcal{R}\}$	$\{\frac{-i}{\pi f}; f \in \mathcal{R}\}$
$\{u_{-1}(t); t \in \mathcal{R}\}$	$\{\frac{1}{2}\delta(f) - \frac{i}{2\pi f}(1-\delta_f); f \in \mathcal{R}\}$
$\{\delta(t-t_0); t \in \mathcal{R}\}$	$\{e^{-i2\pi f t_0}; f \in \mathcal{R}\}$
$\{\delta(at+b); t \in \mathcal{R}\}$	$\{\frac{1}{\|a\|}e^{-i2\pi f \frac{b}{a}}; f \in \mathcal{R}\}$
$\{\delta(t); t \in \mathcal{R}\}$	$\{1; f \in \mathcal{R}\}$
$\{\delta'(t); t \in \mathcal{R}\}$	$\{i2\pi f; f \in \mathcal{R}\}$
$\{\Psi_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT);\ t \in \mathcal{R}\}$	$\{\frac{1}{T}\Psi_{1/T}(f);\ f \in \mathcal{R}\}$
$\{\text{Ш}(t); t \in \mathcal{R}\}$	$\{\text{Ш}(f); f \in \mathcal{R}\}$
$\{\frac{1}{2}(\delta(t+\frac{1}{2}) + \delta(t-\frac{1}{2})); t \in \mathcal{R}\}$	$\{\cos(\pi f); f \in \mathcal{R}\}$
$\{\frac{1}{2}(\delta(t+\frac{1}{2}) - \delta(t-\frac{1}{2})); t \in \mathcal{R}\}$	$\{i\sin(\pi f); f \in \mathcal{R}\}$
$\{\text{sech}(\pi t); t \in \mathcal{R}\}$	$\{\text{sech}(\pi f); f \in \mathcal{R}\}$
$\{J_0(2\pi t); t \in \mathcal{R}\}$	$\begin{cases} \frac{1}{\pi\sqrt{1-f^2}} & f \in \mathcal{R}, \|f\| < 1 \\ 0 & \text{otherwise} \end{cases}$

Table A.1: Continuous Time, Infinite Duration

g	$\mathcal{F}(g)$
$\{\text{sinc}(t); t \in \mathcal{R}\}$	$\{\sqcap(f); f \in \mathcal{R}\}$
$\{\text{sinc}^2(t); t \in \mathcal{R}\}$	$\{\wedge(f); f \in \mathcal{R}\}$
$\{\frac{1}{\pi t}; t \in \mathcal{R}\}$	$\{-i\text{sgn}(f); f \in \mathcal{R}\}$
$\{e^{-i2\pi f_0 t}; t \in \mathcal{R}\}$	$\{\delta(f + f_0); f \in \mathcal{R}\}$
$\{1; t \in \mathcal{R}\}$	$\{\delta(f); f \in \mathcal{R}\}$
$\{\cos(\pi t); t \in \mathcal{R}\}$	$\{II(f); f \in \mathcal{R}\}$
$\{\sin(\pi t); t \in \mathcal{R}\}$	$\{iI_I(f); f \in \mathcal{R}\}$
$\{\frac{1}{2}\delta(t) + \frac{i}{2\pi t}(1 - \delta_t); t \in \mathcal{R}\}$	$\{u_{-1}(f); f \in \mathcal{R}\}$

Table A.2: Continuous Time, Infinite Duration (Duals)

g	$\mathcal{F}(g)$
$\{\sqcap(t);\ t \in [-\frac{T}{2}, \frac{T}{2})\}$, $T \geq 1$	$\{\text{sinc}(f); f \in \{\frac{k}{T};\ k \in \mathcal{Z}\}\}$
$\{\wedge(t); t \in [-\frac{T}{2}, \frac{T}{2})\}$, $T \geq 2$	$\{\text{sinc}^2(f); f \in \{\frac{k}{T};\ k \in \mathcal{Z}\}\}$
$\{\delta(t); t \in [-\frac{T}{2}, \frac{T}{2})\}$	$\{1;\ f \in \{\frac{k}{T};\ k \in \mathcal{Z}\}\}$
$\{II(t); t \in [-\frac{T}{2}, \frac{T}{2})\}$, $T > 1$	$\{\cos(\pi f);\ f \in \{\frac{k}{T};\ k \in \mathcal{Z}\}\}$
$\{I_I(t); t \in [-\frac{T}{2}, \frac{T}{2})\}$, $T > 1$	$\{i\sin(\pi f);\ f \in \{\frac{k}{T};\ k \in \mathcal{Z}\}\}$
$\{t;\ t \in [0, 1)\}$	$\{\frac{1}{2}\delta_k + \frac{i}{2\pi k}(1 - \delta_k);\ k \in \mathcal{Z}\}$

Table A.3: Continuous Time, Finite Duration

g	$\mathcal{F}(g)$
$\{r^n u_{-1}(n);\ n \in \mathcal{Z}\}$ $(\|r\| < 1)$	$\{\frac{1}{1-re^{-i2\pi f}};\ f \in [-\frac{1}{2}, \frac{1}{2})\}$
$\{\Box_N(n);\ n \in \mathcal{Z}\}$	$\{\frac{\sin(2\pi f(N+\frac{1}{2}))}{\sin(\pi f)};\ f \in [-\frac{1}{2}, \frac{1}{2})\}$
$\{\mathrm{sgn}(n);\ n \in \mathcal{Z}\}$	$\{\frac{1-e^{i2\pi f}}{1-\cos 2\pi f};\ f \in [-\frac{1}{2}, \frac{1}{2})\}$
$\{\delta_{n-n_0}; n \in \mathcal{Z}\}$	$\{e^{-i2\pi f n_0};\ f \in [-\frac{1}{2}, \frac{1}{2})\}$
$\{\delta_n;\ n \in \mathcal{Z}\}$	$\{1; f \in [-\frac{1}{2}, \frac{1}{2})\}$
$\{e^{-i2\pi \frac{k}{N} n};\ n \in \mathcal{Z}\}$	$\{\delta(f + \frac{k}{N});\ f \in [-\frac{1}{2}, \frac{1}{2})\}$

Table A.4: Discrete Time, Infinite Duration

g	$\mathcal{F}(g)$
$\{r^n;\ n \in \mathcal{Z}_N\}$	$\{\frac{1-r^N}{1-re^{-i2\pi k/N}};\ k \in \mathcal{Z}_N\}$
$\{1;\ n \in \mathcal{Z}_N\}$	$\{N\delta_k;\ k \in \mathcal{Z}_N\}$
$\{\delta_n;\ n \in \mathcal{Z}_N\}$	$\{1;\ k \in \mathcal{Z}_N\}$
$\{\delta_{n-k};\ n \in \mathcal{Z}_N\}$	$\{e^{-i2\pi \frac{k}{N}};\ k \in \mathcal{Z}_N\}$

Table A.5: Discrete Time, Finite Duration (DFT)

Function	Transform
$\{\exp[-\pi(x^2+y^2)];\ x,y \in \mathcal{R}\}$	$\{\exp\left[-\pi\left(f_X^2+f_Y^2\right)\right];\ f_X,f_Y \in \mathcal{R}\}$
$\{\sqcap(x)\sqcap(y);\ x,y \in \mathcal{R}\}$	$\mathrm{sinc}(f_X)\ \mathrm{sinc}(f_Y);\ f_X,f_Y \in \mathcal{R}\}$
$\{\Lambda(x)\,\Lambda(y);\ x,y \in \mathcal{R}\}$	$\{\mathrm{sinc}^2(f_X)\ \mathrm{sinc}^2(f_Y);\ f_X,f_Y \in \mathcal{R}\}$
$\{\delta(x,y);\ x,y \in \mathcal{R}\}$	$\{1;\ f_X,f_Y \in \mathcal{R}\}$
$\{\exp[i\pi(x+y)];\ x,y \in \mathcal{R}\}$	$\{\delta(f_X-1/2,f_Y-1/2);\ f_X,f_Y \in \mathcal{R}\}$
$\{\mathrm{sgn}(x)\,\mathrm{sgn}(y);\ x,y \in \mathcal{R}\}$	$\{\frac{1}{i\pi f_X}\ \frac{1}{i\pi f_Y};\ f_X,f_Y \in \mathcal{R}\}$
$\{\mathrm{comb}(x)\,\mathrm{comb}(y);\ x,y \in \mathcal{R}\}$	$\mathrm{comb}(f_X)\,\mathrm{comb}(f_Y);\ f_X,f_Y \in \mathcal{R}\}$
$\{\exp[i\pi(x^2+y^2)];\ x,y \in \mathcal{R}\}$	$i\exp\left[-i\pi\left(f_X^2+f_Y^2\right)\right];\ f_X,f_Y \in \mathcal{R}\}$
$\{\exp[-(\lvert x\rvert+\lvert y\rvert)];\ x,y \in \mathcal{R}\}$	$\{\frac{2}{1+(2\pi f_X)^2}\ \frac{2}{1+(2\pi f_Y)^2};\ f_X,f_Y \in \mathcal{R}\}$
$\mathrm{circ}(\sqrt{x^2+y^2};\ x,y \in \mathcal{R}\}$	$\frac{J_1(2\pi\sqrt{f_X^2+f_Y^2})}{\sqrt{f_X^2+f_Y^2}};\ f_X,f_Y \in \mathcal{R}\}$
$\delta(\sqrt{x^2+y+2};\ x,y \in \mathcal{R}\}$	$2\pi r_0 J_0(2\pi r_0\sqrt{f_X^2+f_Y^2});\ f_X,f_Y \in \mathcal{R}\}$

Table A.6: Two-dimensional Fourier transform pairs.

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