11.10 Tables of Transforms

Table I. Fourier Cosine Transforms

See (2) in Sec. 11.8.

	$f(x)$	$\hat{f}_c(w) = \mathcal{F}_c(f)$
1	$\begin{cases} 1 & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin aw}{w}$
2	$x^{a-1} \quad (0 < a < 1)$	$\sqrt{\frac{2}{\pi}} \frac{\Gamma(a)}{w^a} \cos \frac{a\pi}{2}$ ($\Gamma(a)$ see App. A3.1.)
3	$e^{-ax} \quad (a > 0)$	$\sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2 + w^2}\right)$
4	$e^{-x^2/2}$	$e^{-w^2/2}$
5	$e^{-ax^2} \quad (a > 0)$	$\frac{1}{\sqrt{2a}} e^{-w^2/(4a)}$
6	$x^n e^{-ax} \quad (a > 0)$	$\sqrt{\frac{2}{\pi}} \frac{n!}{(a^2 + w^2)^{n+1}} \operatorname{Re}(a + iw)^{n+1}$ Re = Real part
7	$\begin{cases} \cos x & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$	$\frac{1}{\sqrt{2\pi}} \left[\frac{\sin a(1 - w)}{1 - w} + \frac{\sin a(1 + w)}{1 + w}\right]$
8	$\cos(ax^2) \quad (a > 0)$	$\frac{1}{\sqrt{2a}} \cos\left(\frac{w^2}{4a} - \frac{\pi}{4}\right)$
9	$\sin(ax^2) \quad (a > 0)$	$\frac{1}{\sqrt{2a}} \cos\left(\frac{w^2}{4a} + \frac{\pi}{4}\right)$
10	$\frac{\sin ax}{x} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}} (1 - u(w - a))$ (See Sec. 6.3.)
11	$\frac{e^{-x} \sin x}{x}$	$\frac{1}{\sqrt{2\pi}} \arctan \frac{2}{w^2}$
12	$J_0(ax) \quad (a > 0)$	$\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{a^2 - w^2}} (1 - u(w - a))$ (See Secs. 5.5, 6.3.)

Table II. Fourier Sine Transforms

See (5) in Sec. 11.8.

	$f(x)$	$\hat{f}_s(w) = \mathscr{F}_s(f)$
1	$\begin{cases} 1 & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \left[\frac{1 - \cos aw}{w} \right]$
2	$1/\sqrt{x}$	$1/\sqrt{w}$
3	$1/x^{3/2}$	$2\sqrt{w}$
4	$x^{a-1} \quad (0 < a < 1)$	$\sqrt{\frac{2}{\pi}} \frac{\Gamma(a)}{w^a} \sin \frac{a\pi}{2}$ ($\Gamma(a)$ see App. A3.1.)
5	$e^{-ax} \quad (a > 0)$	$\sqrt{\frac{2}{\pi}} \left(\frac{w}{a^2 + w^2} \right)$
6	$\frac{e^{-ax}}{x} \quad (a > 0)$	$\sqrt{\frac{2}{\pi}} \arctan \frac{w}{a}$
7	$x^n e^{-ax} \quad (a > 0)$	$\sqrt{\frac{2}{\pi}} \frac{n!}{(a^2 + w^2)^{n+1}} \operatorname{Im}(a + iw)^{n+1}$ Im = Imaginary part
8	$xe^{-x^2/2}$	$we^{-w^2/2}$
9	$xe^{-ax^2} \quad (a > 0)$	$\frac{w}{(2a)^{3/2}} e^{-w^2/4a}$
10	$\begin{cases} \sin x & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$	$\frac{1}{\sqrt{2\pi}} \left[\frac{\sin a(1 - w)}{1 - w} - \frac{\sin a(1 + w)}{1 + w} \right]$
11	$\frac{\cos ax}{x} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}} u(w - a)$ (See Sec. 6.3.)
12	$\arctan \frac{2a}{x} \quad (a > 0)$	$\sqrt{2\pi} \frac{\sin aw}{w} e^{-aw}$

Table III. Fourier Transforms

See (6) in Sec. 11.9.

	$f(x)$	$\hat{f}(w) = \mathcal{F}(f)$
1	$\begin{cases} 1 & \text{if } -b < x < b \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin bw}{w}$
2	$\begin{cases} 1 & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{e^{-ibw} - e^{-icw}}{iw\sqrt{2\pi}}$
3	$\dfrac{1}{x^2 + a^2} \quad (a > 0)$	$\sqrt{\dfrac{\pi}{2}}\,\dfrac{e^{-a\lvert w\rvert}}{a}$
4	$\begin{cases} x & \text{if } 0 < x < b \\ 2x - b & \text{if } b < x < 2b \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{-1 + 2e^{ibw} - e^{-2ibw}}{\sqrt{2\pi}w^2}$
5	$\begin{cases} e^{-ax} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \quad (a > 0)$	$\dfrac{1}{\sqrt{2\pi}(a + iw)}$
6	$\begin{cases} e^{ax} & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{e^{(a-iw)c} - e^{(a-iw)b}}{\sqrt{2\pi}(a - iw)}$
7	$\begin{cases} e^{iax} & \text{if } -b < x < b \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin b(w - a)}{w - a}$
8	$\begin{cases} e^{iax} & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{i}{\sqrt{2\pi}}\,\dfrac{e^{ib(a-w)} - e^{ic(a-w)}}{a - w}$
9	$e^{-ax^2} \quad (a > 0)$	$\dfrac{1}{\sqrt{2a}}\,e^{-w^2/4a}$
10	$\dfrac{\sin ax}{x} \quad (a > 0)$	$\sqrt{\dfrac{\pi}{2}} \quad \text{if } \lvert w\rvert < a; \quad 0 \text{ if } \lvert w\rvert > a$

CHAPTER 11 REVIEW QUESTIONS AND PROBLEMS

1. What is a Fourier series? A Fourier cosine series? A half-range expansion? Answer from memory.
2. What are the Euler formulas? By what very important idea did we obtain them?
3. How did we proceed from 2π-periodic to general-periodic functions?
4. Can a discontinuous function have a Fourier series? A Taylor series? Why are such functions of interest to the engineer?
5. What do you know about convergence of a Fourier series? About the Gibbs phenomenon?
6. The output of an ODE can oscillate several times as fast as the input. How come?
7. What is approximation by trigonometric polynomials? What is the minimum square error?
8. What is a Fourier integral? A Fourier sine integral? Give simple examples.
9. What is the Fourier transform? The discrete Fourier transform?
10. What are Sturm–Liouville problems? By what idea are they related to Fourier series?

11–20 FOURIER SERIES.

In Probs. 11, 13, 16, 20 find the Fourier series of $f(x)$ as given over one period and sketch $f(x)$ and partial sums. In Probs. 12, 14, 15, 17–19 give answers, with reasons. Show your work detail.

11. $f(x) = \begin{cases} 0 & \text{if} \quad -2 < x < 0 \\ 2 & \text{if} \quad 0 < x < 2 \end{cases}$
12. Why does the series in Prob. 11 have no cosine terms?
13. $f(x) = \begin{cases} 0 & \text{if} \quad -1 < x < 0 \\ x & \text{if} \quad 0 < x < 1 \end{cases}$
14. What function does the series of the cosine terms in Prob. 13 represent? The series of the sine terms?
15. What function do the series of the cosine terms and the series of the sine terms in the Fourier series of $e^x \ (-5 < x < 5)$ represent?
16. $f(x) = |x| \quad (-\pi < x < \pi)$
17. Find a Fourier series from which you can conclude that $1 - 1/3 + 1/5 - 1/7 + - \cdots = \pi/4$.
18. What function and series do you obtain in Prob. 16 by (termwise) differentiation?
19. Find the half-range expansions of $f(x) = x \ (0 < x < 1)$.
20. $f(x) = 3x^2 \quad (-\pi < x < \pi)$

21–22 GENERAL SOLUTION

Solve, $y'' + \omega^2 y = r(t)$, where $|\omega| \neq 0, 1, 2, \cdots, r(t)$ is 2π-periodic and

21. $r(t) = 3t^2 \ (-\pi < t < \pi)$
22. $r(t) = |t| \ (-\pi < t < \pi)$

23–25 MINIMUM SQUARE ERROR

23. Compute the minimum square error for $f(x) = x/\pi \ (-\pi < x < \pi)$ and trigonometric polynomials of degree $N = 1, \cdots, 5$.
24. How does the minimum square error change if you multiply $f(x)$ by a constant k?
25. Same task as in Prob. 23, for $f(x) = |x|/\pi \ (-\pi < x < \pi)$. Why is E^* now much smaller (by a factor 100, approximately!)?

26–30 FOURIER INTEGRALS AND TRANSFORMS

Sketch the given function and represent it as indicated. If you have a CAS, graph approximate curves obtained by replacing ∞ with finite limits; also look for Gibbs phenomena.

26. $f(x) = x + 1$ if $0 < x < 1$ and 0 otherwise; by the Fourier sine transform
27. $f(x) = x$ if $0 < x < 1$ and 0 otherwise; by the Fourier integral
28. $f(x) = kx$ if $a < x < b$ and 0 otherwise; by the Fourier transform
29. $f(x) = x$ if $1 < x < a$ and 0 otherwise; by the Fourier cosine transform
30. $f(x) = e^{-2x}$ if $x > 0$ and 0 otherwise; by the Fourier transform

SUMMARY OF CHAPTER 11

Fourier Analysis. Partial Differential Equations (PDEs)

Fourier series concern **periodic functions** $f(x)$ of period $p = 2L$, that is, by definition $f(x + p) = f(x)$ for all x and some fixed $p > 0$; thus, $f(x + np) = f(x)$ for any integer n. These series are of the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right) \tag{1}$$ (Sec. 11.2)

with coefficients, called the **Fourier coefficients** of $f(x)$, given by the Euler formulas (Sec. 11.2)

$$(2) \qquad \begin{aligned} a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x)\, dx, \qquad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx \end{aligned}$$

where $n = 1, 2, \cdots$. For period 2π we simply have (Sec. 11.1)

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1*}$$

with the *Fourier coefficients* of $f(x)$ (Sec. 11.1)

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx.$$

Fourier series are fundamental in connection with periodic phenomena, particularly in models involving differential equations (Sec. 11.3, Chap, 12). If $f(x)$ is even $[f(-x) = f(x)]$ or odd $[f(-x) = -f(x)]$, they reduce to **Fourier cosine** or **Fourier sine series**, respectively (Sec. 11.2). If $f(x)$ is given for $0 \leqq x \leqq L$ only, it has two **half-range expansions** of period $2L$, namely, a cosine and a sine series (Sec. 11.2).

The set of cosine and sine functions in (1) is called the **trigonometric system**. Its most basic property is its **orthogonality** on an interval of length $2L$; that is, for all integers m and $n \neq m$ we have

$$\int_{-L}^{L} \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L}\, dx = 0, \qquad \int_{-L}^{L} \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L}\, dx = 0$$

and for all integers m and n,

$$\int_{-L}^{L} \cos \frac{m\pi x}{L} \sin \frac{n\pi x}{L}\, dx = 0.$$

This orthogonality was crucial in deriving the Euler formulas (2).

Partial sums of Fourier series minimize the **square error** (Sec. 11.4).

Replacing the trigonometric system in (1) by other orthogonal systems first leads to ***Sturm–Liouville problems*** (Sec. 11.5), which are boundary value problems for ODEs. These problems are ***eigenvalue problems*** and as such involve a parameter λ that is often related to frequencies and energies. The solutions to Sturm–Liouville problems are called ***eigenfunctions***. Similar considerations lead to other orthogonal series such as ***Fourier–Legendre series*** and ***Fourier–Bessel series*** classified as ***generalized Fourier series*** (Sec. 11.6).

Ideas and techniques of Fourier series extend to nonperiodic functions $f(x)$ defined on the entire real line; this leads to the **Fourier integral**

$$f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx]\,dw \qquad \text{(Sec. 11.7)} \tag{3}$$

where

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv, \qquad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv \tag{4}$$

or, in complex form (Sec. 11.9),

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwx}\,dw \qquad (i = \sqrt{-1}) \tag{5}$$

where

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx. \tag{6}$$

Formula (6) transforms $f(x)$ into its **Fourier transform** $\hat{f}(w)$, and (5) is the inverse transform.

Related to this are the **Fourier cosine transform** (Sec. 11.8)

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx\,dx \tag{7}$$

and the **Fourier sine transform** (Sec. 11.8)

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin wx\,dx. \tag{8}$$

The ***discrete Fourier transform (DFT)*** and a practical method of computing it, called the ***fast Fourier transform (FFT)***, are discussed in Sec. 11.9.